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THE EFFECT OF SMALL RESISTIVE FORCES ON RELATIVE EQUILIBRIUM†

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The relative equilibria of a material point in a uniformly rotating frame of reference are investigated. Along with potential forces, which depend on the relative coordinates, and forces of inertia, allowance is made for small resistive forces that depend on the absolute velocity of the point. These forces cause, first, a shift of the equilibrium and, second, a change in the characteristic exponents compared with the conservative case. Necessary and sufficient conditions are established for asymptotic stability. These conditions hold, in particular, in systems that are stable in the first approximation when there is small viscous friction. For the first time, asymptotically stable cases are established for triangular libration points in the classical three-body problem taking into account the resistance of the medium.

1. THE SHIFT OF EQUILIBRIUM IN A RESISTANT MEDIUM

The solution of several problems in theoretical and celestial mechanics involves investigating the equilibrium of a material point relative to a frame of reference rotating at a constant angular velocity ω about a fixed axis OZ. The force function of the system $U=U(x, y, z)$ depends only on the relative coordinates. The equations of motion in vacuum (assuming a unit mass) are

$$\ddot{\mathbf{r}} = \operatorname{grad} U + \mathbf{P}_e + \mathbf{P}_c, \quad \mathbf{r} = (x, y, z)^T, \quad \mathbf{P}_e = \omega^2 \mathbf{r} - (\boldsymbol{\omega}, \mathbf{r})\boldsymbol{\omega}$$
$$\mathbf{P}_c = -2\boldsymbol{\omega} \times \mathbf{r}, \quad \boldsymbol{\omega} = (0, 0, \omega) \tag{1.1}$$

where $\mathbf{P}_e$ and $\mathbf{P}_c$ are the translational and Coriolis forces of inertia [1].

In a position of relative equilibrium we have $\mathbf{r} = \mathbf{r}_0$, $\dot{\mathbf{r}} = \ddot{\mathbf{r}} = 0$, so that

$$\operatorname{grad} U'|_{\mathbf{r}=\mathbf{r}_0} = 0, \quad U' = U + \tfrac{1}{2}\omega^2(x^2 + y^2) \tag{1.2}$$

Together with system (1.1) we shall also consider a more realistic model, in which allowance is made for a small resistive force exerted by the medium, in the direction opposite to that of the absolute velocity $\mathbf{V}$ of the point

$$\ddot{\mathbf{r}} = \operatorname{grad} U + \mathbf{P}_e + \mathbf{P}_c + \mathbf{S}$$
$$\mathbf{S} = -f(V)\mathbf{V}, \quad \mathbf{V} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}, \quad f(V_0) = \varepsilon \ll \omega, \quad \mathbf{V}_0 = \boldsymbol{\omega} \times \mathbf{r}_0 \tag{1.3}$$

Below we present a comparative analysis of system (1.1) and (1.3) in order to establish the existence of points of equilibrium and stability.

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To determine the position of equilibrium of system (1.3), we set $\mathbf{r}=\mathbf{r}^*=\mathbf{r}_0+\Delta\mathbf{r}$, $\dot{\mathbf{r}}=\ddot{\mathbf{r}}=\mathbf{0}$, obtaining

$$\operatorname{grad} U'|_{\mathbf{r}=\mathbf{r}^*}=f(V^*)\mathbf{V}^*,\quad \mathbf{V}^*=\boldsymbol{\omega}\times\mathbf{r}^* \tag{1.4}$$

When there is no resistance, $f=0$, and system (1.4) is satisfied when $\Delta\mathbf{r}=\mathbf{0}$ (i.e. $\mathbf{r}^*=\mathbf{r}_0$). This solution is unique if the second variation of the varied force function

$$U_2=\|\delta^2U'/\delta\mathbf{r}^2\|_{\mathbf{r}=\mathbf{r}_0}$$

is non-degenerate, in which case the position of equilibrium (1.2) of system (1.1) is isolated. In that case system (1.4) has a unique solution in the neighbourhood of $\mathbf{r}_0$

$$\Delta\mathbf{r}=\varepsilon U_2^{-1}(\boldsymbol{\omega}\times\mathbf{r}_0)+O(\varepsilon^2) \tag{1.5}$$

Note that if $\mathbf{r}_0$ is on the axis of rotation, the relative equilibrium is also absolute, so that $\Delta\mathbf{r}\equiv\mathbf{0}$.

The following assertion can be proved

Assertion 1. Assume that the Hessian of the function $U+\frac{1}{2}\omega^2(x^2+y^2)$ does not vanish at the position of equilibrium (1.2). Then, if $\varepsilon>0$ is sufficiently small, the resistive forces (1.3) displace the position of equilibrium by the distance $\Delta\mathbf{r}$ given by (1.5).

2. STABILITY IN THE CASE OF PROPORTIONAL RESISTANCE

Let us investigate the stability of the position of equilibrium of system (1.3). We note first that system (1.1) is a generalized conservative system, since the forces of inertia $\mathbf{P}_e$ and $\mathbf{P}_c$ have the structure of linear potential and gyroscopic forces, respectively. A necessary condition for the stability of the system is therefore that the linear system

$$\ddot{\boldsymbol{\rho}}+2\Omega\dot{\boldsymbol{\rho}}-U_2\boldsymbol{\rho}=0;\quad \boldsymbol{\rho}=\mathbf{r}=\mathbf{r}_0,\quad \Omega=\begin{Vmatrix}0&-\omega&0\\ \omega&0&0\\ 0&0&0\end{Vmatrix} \tag{2.1}$$

should have only pure imaginary characteristic exponents λ_j $(j=1-6)$.

Let us linearize system (1.3) in the neighbourhood of the point $\mathbf{r}^*$ (retaining the notation $\boldsymbol{\rho}$ for the difference $\mathbf{r}^*-\mathbf{r}$)

$$\ddot{\boldsymbol{\rho}}+2\Omega\dot{\boldsymbol{\rho}}-U_2^*\boldsymbol{\rho}-d\mathbf{S}=0 \tag{2.2}$$

where U_2^* is the second variation of U' and $d\mathbf{S}$ is the differential of the vector-valued function $\mathbf{S}$ evaluated at $\mathbf{r}^*$. As $\varepsilon\to0$ Eq. (2.2) tends to (2.1), and the continuous dependence of the characteristic exponents on the parameter implies the following.

Assertion 2. A necessary condition for the equilibrium in a medium with small resistance to be stable is that the equilibrium (1.2) in a vacuum should be stable in the first approximation.

To obtain the sufficient conditions for stability in system (2.2), one might at first sight try to apply the Kelvin–Chetayev theorems [2] on the effect of dissipative and gyroscopic forces on the stability of a conservative system. However, formulae (1.3) show that the expression for the dissipative force S in terms of relative coordinates and velocities contains terms of a different structure.

Consider, in particular, the case when $f=\text{const}$, which is typical for forces of viscous friction. In that case

$$\mathbf{S} = -\varepsilon\mathbf{V}, \quad d\mathbf{S} = -\varepsilon(\dot{\boldsymbol{\rho}} + \Omega\boldsymbol{\rho}) \tag{2.3}$$

The first term in $d\mathbf{S}$ has the form of linear dissipative forces, and the second represents positional non-conservative forces. It was observed in [3], when studying uniform rotations of a rigid body, that forces of dissipative structure in an absolute reference system may have a positional component in the attached system.

Theorem 1. In the case of proportional resistance (2.3), a sufficient condition for the equilibrium position of system (1.3) to be stable is that the characteristic exponents of system (2.1) should be pure imaginary and different from one another.

Proof. The characteristic equation of system (2.2) is

$$F(\lambda, \varepsilon) = \det\|(\lambda^2 + \varepsilon\lambda)E_3 + (2\lambda + \varepsilon)\Omega - \overset{*}{U_2}\| = 0 \tag{2.4}$$

Let λ_j^* $(j = 1–6)$ denote the roots of the equation

$$\det\|\lambda^2 E_3 + 2\lambda\Omega - \overset{*}{U_2}\| = 0$$

By assumption, if ε is sufficiently small, then $\operatorname{Re}\lambda_j^* = 0$ and $\lambda_k^* \neq \lambda_j^*$ for $k \neq j$ [3]. In view of the structure of $F(\lambda, \varepsilon)$ as shown in (2.4), we obtain

$$F(\overset{*}{\lambda_j}, 0) = 0, \quad \partial F(\overset{*}{\lambda_j}, 0)/\partial\lambda = 2\partial F(\overset{*}{\lambda_j}, 0)/\partial\varepsilon \neq 0$$

Hence the roots of Eq. (2.4) may be written as

$$\lambda_j(\varepsilon) = \overset{*}{\lambda_j} - \varepsilon/2 + O(\varepsilon^2) \tag{2.5}$$

When $\varepsilon > 0$ we have $\operatorname{Re}\lambda_j < 0$ in (2.3), implying the desired conclusion about asymptotic stability in the first approximation.

Note that the theorem does not require the unperturbed system (1.1) to be stable in Lyapunov's sense.

3. STABILITY CONDITIONS FOR NON-LINEAR RESISTANCE LAWS

We will now investigate an arbitrary law of resistance, assuming that the function f in (1.3) is positive and differentiable. Linearization of system (2.1) produces additional terms in formula (3.1). Since

$$\mathbf{S} - \mathbf{S}^* = f(V^*)\mathbf{V}^* - f(V)\mathbf{V} = f(V^*)(\mathbf{V}^* - \mathbf{V}) + f'(V^*)(V^* - V)\mathbf{V}^* + o(V^* - V) =$$

$$= -\varepsilon(\dot{\boldsymbol{\rho}} + \boldsymbol{\omega} \times \boldsymbol{\rho}) + f'(V_0)[(\boldsymbol{\omega}, \mathbf{r}_0)(\boldsymbol{\omega}, \boldsymbol{\rho}) - \omega^2(\mathbf{r}_0, \boldsymbol{\rho}) - (\mathbf{V}_0, \dot{\boldsymbol{\rho}})]\mathbf{V}_0 / V_0 + \ldots$$

(omitting terms non-linear in $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$, as well as quantities of order $O(\varepsilon^2)$), it follows that if $f'(\upsilon_0) \neq 0$ the expression for $d\mathbf{S}$ in (2.2) will have the following form, provided the origin is suitably chosen (so that $(\omega, \mathbf{r}_0) = 0$)

$$d\mathbf{S} = -\varepsilon(D\dot{\boldsymbol{\rho}} + K\boldsymbol{\rho} + N\boldsymbol{\rho}) \tag{3.1}$$

$$D = E + \frac{2\gamma}{r_0^2}\begin{Vmatrix} y_0^2 & -x_0 y_0 & 0 \\ -x_0 y_0 & x_0^2 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad K = \frac{\gamma}{r_0^2}\begin{Vmatrix} -2x_0 y_0 & x_0^2 - y_0^2 & 0 \\ x_0^2 - y_0^2 & 2x_0 y_0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

$$N=(1+\gamma)\Omega, \quad \gamma=\frac{V_0 f'(V_0)}{2f(V_0)}, \quad r_0^2=x_0^2+y_0^2$$

The parameter γ is determined by the type of resistance ($\gamma=0$ for proportional resistance, $\gamma=\frac{1}{2}$ for a square law, $\gamma=-\frac{1}{2}$ for resistance independent of the absolute value of the velocity, etc.).

The first term in (3.1) has the structure of linear dissipative forces (total dissipation if $\gamma>-\frac{1}{2}$), the second, of potential forces, and the third, of non-conservative positional forces.

Theorem 2. Assume that the matrix U_2 is negative definite and that γ lies in the interval $(-\frac{1}{3}, 1)$. Then the equilibrium position of system (1.3) is asymptotically stable, provided ε is sufficiently small.

Proof. The assumptions of Propositions 1 and 2 all hold. In addition, by one of the Kelvin–Chetayev theorems, equilibrium in a vacuum is stable in Lyapunov's sense. Let us construct a Lyapunov function for the system with friction (3.1)

$$L=(\mathbf{\rho},(\varepsilon K-U_2^*)\mathbf{\rho})+(\dot{\mathbf{\rho}},\dot{\mathbf{\rho}})+\varepsilon(1+\gamma)(\dot{\mathbf{\rho}},\mathbf{\rho}) \tag{3.2}$$

This quadratic form is positive definite for sufficiently small ε; its total derivative with respect to time along trajectories of Eqs (2.2) and (3.1) is

$$\dot{L}=\varepsilon[(1+\gamma)(\mathbf{\rho},U_2^*\mathbf{\rho})+(1+\gamma)(\dot{\mathbf{\rho}},\dot{\mathbf{\rho}})-2(\dot{\mathbf{\rho}},D\dot{\mathbf{\rho}})]+o(\varepsilon) \tag{3.3}$$

An easy check will show that, under the above restrictions on the parameter γ, the function (3.3) is negative definite, implying the required asymptotic stability.

Of the two assumptions of Theorem 2, the basic one is the restriction on the matrix U_2, since in real mechanical systems the resistance of the medium is characterized by the coefficient $\gamma\in[0, \frac{1}{2}]$. If U_2 is positive definite or has one positive eigenvalue and two negative ones, the necessary conditions for stability fail to hold. In the case of two positive and one negative eigenvalues, the assumptions of Theorem 2 do not hold, but those of Assertion 2 may be satisfied (gyro-stabilization). This case will be considered below.

Lemma. Let $F(\lambda, \varepsilon)$ be a polynomial of even degree in λ with real coefficients which are continuously differentiable with respect to a parameter ε. Assume that the equation $F(\lambda, \varepsilon)=0$ has only pure imaginary roots when $\varepsilon=0$. Then the real parts of the roots when $\varepsilon\neq 0$ are given by the formula

$$\operatorname{Re}\lambda(\varepsilon)=\varepsilon\frac{F_\varepsilon(-\lambda,0)-F_\varepsilon(\lambda,0)}{2F_\lambda(\lambda,0)}+o(\varepsilon) \tag{3.4}$$

(the subscript indicates partial differentiation).

This assertion follows from the theorem of implicit functions and the fact that $F(\lambda, 0)$ is an even function.

We shall use formula (3.4) to analyse the asymptotic stability of the origin in system (2.2). The characteristic equation, in the notation of formula (3.1), is

$$F(\lambda, \varepsilon)=\det\|\lambda^2E_3+2\lambda\Omega-U_2^*+\varepsilon(D\lambda+K+N)\|=0 \tag{3.5}$$

If the assumptions of Assertion 2 are satisfied, we can use the lemma to estimate the roots of Eq. (3.5) and determine ermine whether they lie in the left half-plane.

We will first consider an important special case in which this test is considerably simplified.

Theorem 3. Assume that (1) the matrix U_2 has an eigenvector parallel to the axis of rotation OZ; (2) the corresponding eigenvalue a_3 is negative; (3) the vector $\mathbf{r}_0$ is orthogonal to OZ.

Then a sufficient condition for asymptotic stability is that

$$(1+\gamma)D^{1/2} > |\gamma| \left| \frac{a_1+a_2}{2} - \frac{(U_2\mathbf{r}_0, \mathbf{r}_0)}{r_0^2} \right| \tag{3.6}$$

$$D = [2\omega^2 - {}^1\!/\!{}_2(a_1+a_2)]^2 - a_1 a_2 > 0, \quad a_1 a_2 > 0, \quad 2\omega^2 - {}^1\!/\!{}_2(a_1+a_2) > 0$$

where a_1 and a_2 are the two other eigenvalues of U_2. If at least one of inequalities (3.6) is true with reversed sign, the equilibrium is unstable.

Proof. It is obvious that the third condition may always be ensured by suitable choice of the origin. In addition, a rotation of the X and Y axes will ensure that U_2 will have a diagonal form $U_2 = \text{diag}\{a_1, \ a_2, \ a_3\}$. The characteristic polynomial (3.6) factorizes into the product of a trinomial $\lambda^2 + \varepsilon\lambda - a_3$ whose roots lie in the left half-plane when $\varepsilon > 0$ is small and a fourth-degree polynomial

$$F_1(\lambda, \varepsilon) = \det \begin{Vmatrix} \lambda^2 + \varepsilon\lambda - a_1 + \gamma^*(\lambda y^2 - \omega xy) & -(2\lambda+\varepsilon)\omega - \gamma^*(\omega y^2 + \lambda xy) \\ (2\lambda+\varepsilon)\omega + \gamma^*(\omega x^2 - \lambda xy) & \lambda^2 + \varepsilon\lambda - a_2 + \gamma^*(\lambda x^2 + \omega xy) \end{Vmatrix} \tag{3.7}$$

$$\gamma^* = 2\varepsilon\gamma / r_0^2$$

When $\varepsilon = 0$ we have

$$F_1(\lambda, 0) = (\lambda^2 - a_1)(\lambda^2 - a_2) + 4\lambda^2\omega^2 \tag{3.8}$$

The roots of this polynomial are purely imaginary and pairwise distinct if and only if the second group of inequalities (3.6) holds, in which case

$$\lambda_{1,2}^2(0) = {}^1\!/\!{}_2(a_1+a_2) - 2\omega^2 \pm D^{1/2} \tag{3.9}$$

Substituting this expression into (3.4), we get

$$-1 - \gamma + \gamma \frac{(a_1+a_2)r_0^2 - 2(U_2\mathbf{r}_0, \mathbf{r}_0)}{\pm 2 r_0^2 D^{1/2}} < 0$$

which is equivalent to the first inequality of (3.6).

Corollary. If U_2 is negative definite and the first assumption of the theorem holds, then the equilibrium position is asymptotically stable for all $\gamma > -1$.

Indeed, in that case the right-hand side of inequality (3.6) is at most $|\gamma(a_1 - a_2)|/2$, while $D \geqslant (a_1 - a_2)^2/4$. When $\gamma > -1$ inequality (3.6) is surely true.

This result completes the conclusions of Theorem 2.

4. TRIANGULAR LIBRATION POINTS IN A RARIFIED MEDIUM

The classical circular restricted three-body problem admits of Lagrange's solutions, which are also known as triangular points of libration (see [5]). Corresponding to these solutions are relative equilibrium points of a material point in a frame of reference rotating together with the main attracting bodies. The equations of motion in a vacuum have the form of (1.1), where μ is the mass ratio of the main bodies

$$\omega = 1, \quad r_0 = \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}, 0\right)^T, \quad U_2 = \begin{Vmatrix} 3/4 & k & 0 \\ k & 9/4 & 0 \\ 0 & 0 & -1 \end{Vmatrix}, \quad k = \frac{3\sqrt{3}}{4}(1-2\mu)$$

The conditions for stability in the first approximation (the first group of inequalities (3.6)) are satisfied in the range $0<\mu<\mu^*=0.03852\ldots$. The low resistance of the medium affects the motion of the three bodies to different degrees: the resistance is proportional to the square of the diameter and the mass to the cube of the diameter; hence the acceleration is inversely proportional to the diameter. In the restricted formulation, therefore, we may disregard the influence on the motion of the main bodies.

The displacement of the point of libration L_4 is determined by formula (1.5)

$$\Delta \mathbf{r} = \varepsilon U_2^{-1}\mathbf{r}_0 = \varepsilon(-3\sqrt{3}\,(1-\mu+\mu^2),\, 3(1-2\mu),\, 0)(27/8-2k^2)^{-1} \tag{4.1}$$

Figure 1 illustrates the following: the effect of friction is to shift the point of libration in the direction of the rotation; since the angle between the vectors $\mathbf{r}_0$ and $\Delta\mathbf{r}$ is obtuse, we have $|r^*|<|r_0|$.

Theorems 1 and 3 imply that the shifted point of libration is asymptotically stable for certain laws of resistance. The result is quite unexpected, since as yet the only known stability results in the three-dimensional problem, when there is no resistance, are stability for the majority of the initial conditions and formal stability in the non-resonant case, and also instability for resonances of the third and fourth order [5]. The stabilizing nature of the forces of friction, which are intrinsically dissipative, is conditioned here by the existence of the "gratuitous" source of energy provided by the main bodies, appearing as non-conservative positional forces.

The first condition of (3.6) is equivalent to a single inequality

$$\frac{\gamma+1}{|\gamma|} > \left|\frac{1}{2} - \frac{216-17(4D-1)}{108+4(4D-1)}\right| D^{-1/2}, \quad D = \frac{1-27\mu(1-\mu)}{4} \in \left(0, \frac{1}{4}\right) \tag{4.2}$$

The domain of asymptotic stability in the plane of the parameters μ and γ is shown in Fig. 2. In particular, in the case of a proportional resistance law $(\gamma=0)$ the shifted libration point is asymptotically stable for all $\mu\in(0,\ \mu^*)$; for the case of a square law $(\gamma=1/2)$ or resistance proportional to the square root of the velocity $(\gamma=-1/4)$, it is unstable. In the intermediate case, there is a range of asymptotic stability $\mu\in(0,\ \mu(\gamma))$.

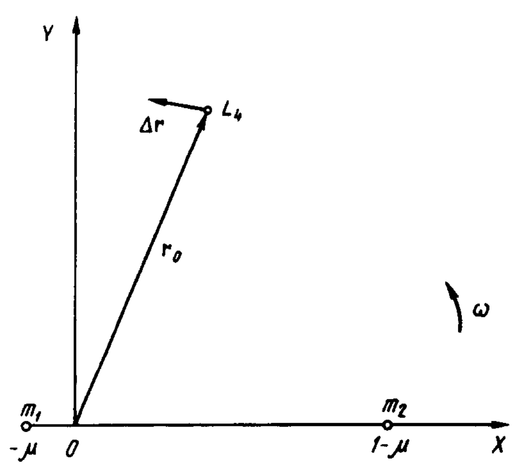

Fig. 1.

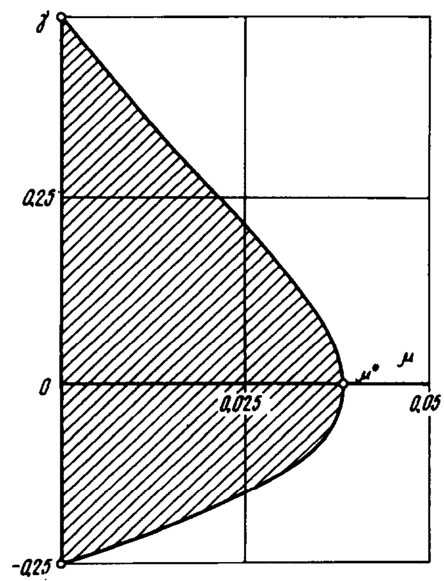

Fig. 2.

5. RELATIVE EQUILIBRIUM OF A NON-FREE PARTICLE

The above results may be extended to the case of a particle moving over a surface

$$z = \varphi(x, y) \tag{5.1}$$

Assuming that the constraint (5.1) is ideal, let us set up the equations of relative motion in Lagrangian form. We have expressions for the kinetic energy T, the new force function U_p and the generalized forces Q in terms of the generalized coordinates x, y and their derivatives

$$\begin{aligned} T &= \tfrac{1}{2}V^2 = \tfrac{1}{2}[(\varphi_x^2+1)\dot{x}^2 + 2\varphi_x\varphi_y\dot{x}\dot{y} + (\varphi_y^2+1)\dot{y}^2] + \omega(x\dot{y} - y\dot{x}) + \\ &+ \tfrac{1}{2}\omega^2(x^2+y^2), \quad U_p(x,y) = U(x,y,\varphi(x,y)) \\ Q_x &= S_x + \varphi_x S_z, \quad Q_y = S_y + \varphi_y S_z \end{aligned} \tag{5.2}$$

where φ_x, φ_y are the partial derivatives of the function (5.1) and S_x, S_y, S_z the components of the resistance vector $\mathbf{S}$.

The Lagrange equations of the second kind, in matrix notation, are

$$\begin{aligned} &B\ddot{\mathbf{r}} + (\dot{x}B_x + \dot{y}B_y)\dot{\mathbf{r}} - \tfrac{1}{2}\operatorname{grad}(B\dot{\mathbf{r}}, \dot{\mathbf{r}}) + 2\Omega\dot{\mathbf{r}} - \operatorname{grad} U' = \mathbf{Q} \\ &U' = U_p + \tfrac{1}{2}\omega^2 r^2, \quad \mathbf{r} = (x, y) \\ &\Omega = \begin{Vmatrix} 0 & -\omega \\ \omega & 0 \end{Vmatrix}, \quad B = \begin{Vmatrix} 1+\varphi_x^2 & \varphi_x\varphi_y \\ \varphi_x\varphi_y & 1+\varphi_y^2 \end{Vmatrix} \end{aligned} \tag{5.3}$$

where B_x and B_y denote the partial derivatives of B.

If $\mathbf{S} \equiv 0$, the equilibrium position $\mathbf{r}_0$ is determined from the condition (1.2) for U' to be stationary. The presence of a resistance (1.3) in formulae (5.2) gives

$$\mathbf{Q} = -f(V)(B\dot{\mathbf{r}} + \Omega\mathbf{r}) \tag{5.4}$$

By analogy with (1.5), one obtains the following formula for the displaced equilibrium under the action of a resistive force (5.4)

$$\Delta\mathbf{r} = \varepsilon U_2^{-1}\Omega\mathbf{r}_0 + O(\varepsilon^2) \tag{5.5}$$

This formula holds provided that the Hessian of the function $U_p + \tfrac{1}{2}\omega^2(x^2+y^2)$ does not vanish at $\mathbf{r} = \mathbf{r}_0$.

When checking for stability in the first approximation, one ignores the second and third terms in Eq. (5.3). If $\mathbf{S} \equiv \mathbf{0}$, the characteristic equation is

$$\det\|B\lambda^2 + 2\Omega\lambda - U_2\| = 0 \tag{5.6}$$

For a proportional law of friction $\mathbf{S} = -\varepsilon\mathbf{V}$ the characteristic polynomial is similar in form to (2.4)

$$F(\varepsilon, \lambda) = \det\|(\lambda^2 + \varepsilon\lambda)B + (2\lambda + \varepsilon)\Omega - U_2^*\| = 0$$

Repeating the proof of Theorem 1, we obtain the following theorem.

Theorem 1′. If Eq. (5.6) has only simple purely imaginary roots and the proportional resistance if sufficiently small, the shifted position of equilibrium of the particle on the surface (5.1) is asymptotically stable.

For an arbitrary law of resistance, one can apply the method used to prove Theorem 3, since in that case Eq. (5.6) is biquadratic. The result is as follows.

Theorem 3′. If Eq. (5.6) has only simple purely imaginary roots and the resistance is characterized by a parameter γ, the asymptotic stability criterion is

$$(1+\gamma)D^{1/2} > |\gamma| \left| \frac{a_1+a_2}{2} - \frac{(B^{-1}U_2\mathbf{r}_0, \mathbf{r}_0)}{r_0^2} \right| \tag{5.7}$$

$$D = [2\omega^2/\Delta_B - (a_1+a_2)/2]^2 - a_1 a_2$$

where $a_{1,2}$ are the eigenvalues of the matrix $B^{-1}U_2$, $\Delta_B = \det \| B \|$.

Corollary. If U_2 is negative definite, the equilibrium position is asymptotically stable for all $\gamma > -1$.

The proof is similar to the proof of the corollary to Theorem 3.

Example ("centrifuge"). A heavy sphere is moving over a surface that is rotating at a constant angular velocity ω about the Z axis, as described in relative coordinates by the equation

$$z = \varphi(x, y) = x^2 + \alpha x^3 + \beta y^2, \quad \alpha > 0, \quad \beta > 0$$

Putting the acceleration due to gravity equal to unity, we write the condition of equilibrium in a vacuum as

$$\varphi_x = \omega^2 x, \quad \varphi_y = \omega^2 y \tag{5.8}$$

System (5.8) has an isolated solution

$$x_0 = (\omega^2 - 2)/3\alpha, \quad y_0 = 0 \tag{5.9}$$

Let us assume that the sphere experience a resistive force due to the air, proportional to the square of the velocity ($\gamma = 1/2$). Then the shift of the equilibrium may be computed by formula (5.5), where $U_2 = \operatorname{diag}\{2 = \omega^2, \omega^2 - 2\beta\}$. The matrix U_2 is invertible if $\omega^2 \neq 2$ and $\omega^2 \neq 2\beta$, in which case

$$\Delta\mathbf{r} = \varepsilon\left(0, \frac{\omega(\omega^2-2)}{3\alpha(\omega^2-2\beta)}\right) + o(\varepsilon)$$

Consequently, in the first approximation the shift occurs in a plane $x = \text{const}$, and moreover, if $\det U_2 > 0$, it occurs in the direction opposite that of the rotation; otherwise the direction is that of the rotation.

The necessary condition for stability holds only in the first of these two cases, and then, if

$$\beta > \omega^2/2 > 1 \tag{5.10}$$

then by Theorem 3′ the shifted equilibrium is asymptotically stable.

If the double inequality (5.10) is true in the opposite sense, the necessary conditions for stability are

$$\beta > 1 - 2\omega^2, \quad (\beta - 1 + 2\omega^2)^2 > (\omega^2 - 2)(2\beta - \omega^2) \tag{5.11}$$

We have expressions for the matrix B and the eigenvalues $a_{1,2}$

$$B = \operatorname{diag}\{1 + \omega^4 x_0^2, 1\}, \quad a_1 = 9\alpha^2(2-\omega^2)[9\alpha^2 + \omega^4(\omega^2-2)^2]^{-1}, \quad a_2 = \omega^2 - 2\beta$$

Figure 3 shows the domain of stability in the plane of the parameters β, ω^2 on the assumption that

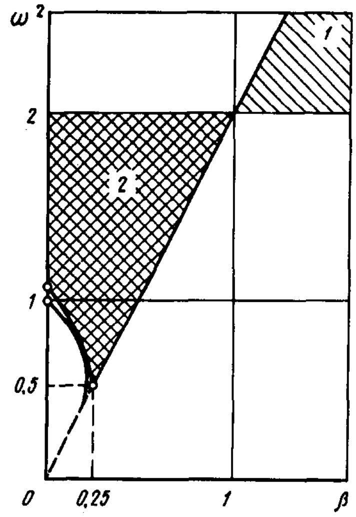

Fig. 3.

$\alpha = 1$. The domain is the union of an infinite sector 1 in which the sufficient conditions (5.10) hold and a curvilinear quadrilateral 2 in which inequalities (5.7) and (5.11) hold.

Note that the equilibrium position (5.8) is stable in a vacuum in sector 1, and in addition the necessary conditions (5.11) for its stability hold in a curvilinear quadrilateral slightly larger than 2, bounded below by the curve $\beta = -1 - \omega^2 + 2\omega(2-\omega^2)^{1/2}$.

We shall discuss one more case of constrained motion: suppose the particle is forced to remain on a curve whose equation in relative coordinates is

$$y = y(x), \quad z = z(x) \tag{5.12}$$

The equation of motion may be set up by analogy with (5.2) and (5.3)

$$B\ddot{x} + \tfrac{1}{2}B'\dot{x}^2 - \omega^2(x + yy') - U_p' = Q \tag{5.13}$$

$$B = 1 + y'^2(x) + z'^2(x), \quad U_p = U(x, y(x), z(x)), \quad Q = S_X + y'S_Y + z'S_Z$$

The condition for equilibrium in a vacuum is

$$U_p' + \omega^2(x + yy') = 0$$

The shift of the position of equilibrium is determined from the formula

$$\Delta x = \varepsilon\omega U_2^{-1}(y_0 y_0' - x_0) + o(\varepsilon), \quad U_2 = U_p''(x_0) + \omega^2(1 + y_0 y_0'' + y_0'^2)$$

which holds provided that $U_2 \neq 0$.

Since system (5.13) has only one degree of freedom, gyrostabilization is impossible, and the position of equilibrium $x = x_0$ is stable if $U_2 < 0$ and unstable if $U_2 > 0$. In the first approximation, the resistance Q is the sum of potential and dissipative linear forces. By the Kelvin–Chetayev theorems, if $\gamma > 1/2$, the stability in a vacuum implies asymptotic stability in a resistive medium. When $U_2 > 0$ the shifted equilibrium is also unstable.

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